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Exercises for 'Functional Analysis 2' [MATH-404]

(26/05/2025)

Ex 14.1 (Quadratic perturbations of convex functionals)
Let $E\colon H \to \mathbb{R}_+ \cup \{\infty\}$ be a convex functional on a real Hilbert space H. Given any $x \in H$ and any $\tau > 0$, we define the convex functional $F_{x,\tau}\colon H \to \mathbb{R}_+ \cup \{\infty\}$ by

$$F_{x,\tau}(y) := E(y) + \frac{\|y-x\|^2}{2\tau}.$$

Show $\mathscr{D}(F_{x,\tau}) = \mathscr{D}(E)$ and for every $y \in H$ that

$$\partial^- F_{x,\tau}(y) = \partial^- E(y) + \frac{y-x}{\tau}.$$

where we use the convention that translates of the empty set are empty.

Ex 14.2 (Uniqueness and fundamental properties of gradient flow trajectories*)
Let $x\colon \mathbb{R}_+ \to H$ be a gradient flow trajectory of a functional E as specified in the lecture notes. Show the following properties.

a) Given any other gradient flow trajectory $y\colon \mathbb{R}_+ \to H$, every $t \in \mathbb{R}_+$ satisfies

$$\big\|x(t) - y(t)\big\| \leq \big\|x(0) - y(0)\big\|.$$

In particular, gradient flow trajectories with fixed initial points are unique.
Hint: Differentiate the assignment $t \mapsto \|x(t) - y(t)\|^2/2$.

b) The assignment $t \mapsto E(x(t))$ is nondecreasing on $\mathbb{R}_+$ and locally Lipschitz continuous on $(0,\infty)$.
Hint: First prove local Lipschitz continuity of x on $(0,\infty)$. To this aim, it suffices to prove $\|\dot{x}\|$ is bounded on each interval of the form $[\varepsilon,\infty)$, where $\varepsilon > 0$. Deduce local Lipschitz continuity of $E \circ x$ using the defining properties of a gradient flow trajectory.

c) For every $z \in H$ and every $t > 0$,

$$E(x_t) \leq E(z) + \frac{\big\|x(0) - z\big\|^2}{2t}.$$

Hint: Use a) and b).

Ex 14.3 (Slope and Laplacian)
Let $E\colon H \to \mathbb{R}_+ \cup \{\infty\}$ be a convex and lower semicontinuous functional with nonempty domain. We define its slope $|\partial^- E|\colon H \to \mathbb{R}_+ \cup \{\infty\}$ through

$$|\partial^- E|(x) := \begin{cases} \sup\limits_{y \in H\setminus\{x\}} \dfrac{(E(y) - E(x))^-}{\|x - y\|} & \text{if } x \in \mathscr{D}(E), \\ \infty & \text{otherwise.} \end{cases}$$

a) Show for every $x \in \mathscr{D}(\partial^- E)$ and every $x^* \in \partial^- E(x)$,

$$|\partial^- E|(x) \leq \|x^*\|.$$

b) Deduce that for every $x \in \mathscr{D}(\partial^- E)$,

$$|\partial^- E|(x) = \min_{x^* \in \partial^- E(x)} \|x^*\|.$$

Hint: Given any $\tau > 0$, consider the unique minimizer $x_\tau \in H$ of the functional $F_{x,\tau}$ from above. You can use without proof that $\|x - x_\tau\|/\tau \leq |\partial^- E|(x)$.

c) Let $x\colon \mathbb{R}_+ \to H$ be a gradient flow trajectory of E. Deduce from b) that for *every* $t > 0$, the right derivative

$$x'^{+}(t) := \lim_{h \to 0+} \frac{x(t+h) - x(t)}{h}$$

exists in H. Moreover, it is equal to minus the unique element of minimal norm in $\partial^- E(x_t)$. The same holds at zero if $x(0) \in \mathscr{D}(\partial^- E)$.
Hint: First clarify why it suffices to prove the claim for $t = 0$ assuming $|\partial^- E|(x(0)) < \infty$. Then show for every $h > 0$ that $\|x(h) - x(0)\|/h \leq |\partial^- E|(x(0))$.

Ex 14.4 (Extension to the closure of the domain)
The goal of this exercise is to prove that gradient flow trajectories can be started from any point in the closure of the domain (and not only the domain, as already covered in the lecture). To this aim, since $o \in \overline{\mathscr{D}(E)}$, fix a sequence $(o_n)_{n\in\mathbb{N}}$ in $\mathscr{D}(E)$ converging to o. We aim to show the corresponding gradient flow trajectories starting at the o_n's converge to a gradient flow trajectory starting at o.

a) Let $x^n\colon \mathbb{R}_+ \to H$ denote the gradient flow trajectory starting at o_n, where $n \in \mathbb{N}$. Show $(x^n)_{n\in\mathbb{N}}$ converges uniformly to a continuous curve $x\colon \mathbb{R}_+ \to H$.
Hint: Use Exercise 14.2.

b) Given any $t_0 > 0$, show the existence of a constant $C(t_0) > 0$ such that for every $n \in \mathbb{N}$,

$$\frac{1}{2}\int_{t_0}^{\infty} \big\|\dot{x}^n(t)\big\|^2 \,\mathrm{d}t \leq C(t_0)$$

Hint: Use Exercise 14.2 to find a uniform upper bound on $E(x^n_{t_0})$. Then apply the energy estimate from the proof of Theorem 5.11 from the lecture notes.

c) Show the curve x from a) is a gradient flow trajectory starting at o.